Dynamic Logic of Product Relation Changers

Ryo Hatano[1] Pimolluck Jirakunkanok[2]
Katsuhiko Sano[3]

[1]Tokyo University of Science, Japan

[2]Independent Researcher, Thailand

[3]Hokkaido University, Japan

DaLi2025 @ Shaanxi Normal University
21st Oct. 2025

State-deletion vs Link-deletion

- Two primary approaches
  for handling agents' knowledge & belief change.
- State-deletion (or domain restriction)
  - **PAL** (Plaza, 1979)
  - **DEL** (Baltag et al., 1998)
- Link-deletion (or accessibility change)
  - **PAL** (Gerbrandy & Groeneveld, 1997)
  - **AUL** & **GAUL** (Kooi & Renne, 2011)
  - **DLRC** (van Benthem & Liu, 2007)
- In this talk, we focus on a link-deletion approach!

Link-deletion Approach

"**PAL**"-style	**AUL** (Kooi & Renne, 2011)	**DLRC** (van Benthem & Liu, 2007)
"**DEL**"-style	**GAUL** (Kooi & Renne, 2011)	?

- **A**rrow **U**pdate **L**ogic restricts a's accessibility to the one from φ-worlds to ψ-worlds.
- **D**ynamic **L**ogic of **R**elation **C**hangers changes a's accessibility by a program in iteration-free **PDL**.
  - Generalizes **AUL**.
- **G**eneralized **AUL** expands **AUL** with action models and has the same expressivity as **DEL**.

Motivation

We integrate the notions of relation changers and action models within a unified framework.

PRC: Dynamic logic of product relation changers

- generalizes both relation changers & (generalized) arrow updates.
- has sound & complete axiomatization
  (inside-out strategy using recursion axioms).
- also has cut-free labelled sequent calculus.

Outline

1. Dynamic Logic of Relation Changers (**DLRC**)
2. Dynamic Logic of Product Relation Changers (**PRC**)
3. Labelled Sequent Calculus for **PRC**

Outline

1. Dynamic Logic of Relation Changers (**DLRC**)
2. Dynamic Logic of Product Relation Changers (**PRC**)
3. Labelled Sequent Calculus for **PRC**

Syntax for **DLRC**

Language $\mathcal{L}^{+}$ for **DLRC**

(= iteration-free **PDL** + relation changers r):

$$\begin{array}{rcl} \mathrm{FORM}_{\mathcal{L}^+} \ni \varphi & ::= & p \mid \neg\varphi \mid (\varphi \to \varphi) \mid [\alpha]\varphi \mid [\mathsf{r}]\varphi, \\ \mathrm{PR}_{\mathcal{L}^+} \ni \alpha & ::= & a \mid (\alpha \cup \alpha) \mid (\alpha;\alpha) \mid ?\varphi, \\ \mathrm{RC}_{\mathcal{L}^+} \ni \mathsf{r} & ::= & (a := \alpha_a)_{a \in \mathbf{AP}}, \end{array}$$

where $p \in \mathbf{P}$ & $a \in \mathbf{AP}$ (atomic programs).

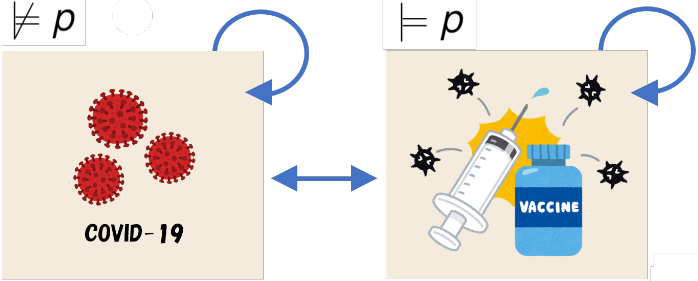
- $[\mathsf{r}]\varphi$: "After changing an accessibility relation for each program a by α_a, φ holds."
- We also use $\mathsf{r}(a)$ to mean α_a.

Introspective Ann. $[r_{!p}]$ (van Ditmarsch et al. 2007)

Let p be "COVID-19 vaccine is available."

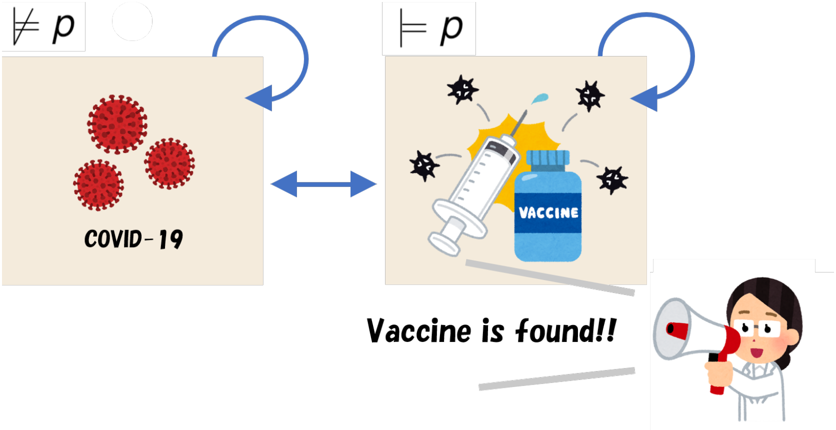
$\not\models p$

$\models p$



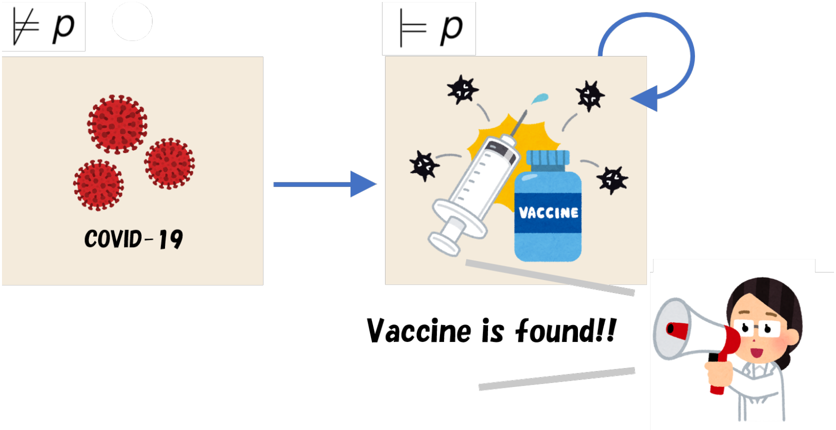
Introspective Ann. $[\mathrm{r}_{!p}]$ (van Ditmarsch et al. 2007)

Let p be "COVID-19 vaccine is available."



Introspective Ann. $[\mathsf{r}_{!p}]$ (van Ditmarsch et al. 2007)

Let p be “COVID-19 vaccine is available.”



$$\mathsf{r}_{!p} = (a := a; ?p)_{a \in \mathbf{AP}}$$

Other Examples of Relation Changers

Atomic Programs Exchange: Let **AP** = $\{a, b, c\}$.

$$\mathsf{r} := (a := b, b := c, c := a).$$

Preference Upgrade:

$$\mathsf{r}_{\sharp\varphi} := (a := (?\neg\varphi; a) \cup (a; ?\varphi))_{a \in \mathbf{AP}}$$

Arrow Updates: Let $U \subseteq \mathrm{FORM}_{\mathcal{L}} \times \mathbf{AP} \times \mathrm{FORM}_{\mathcal{L}}$.

$$\mathsf{r}_{*U} := \left(a := \bigcup_{(\varphi, a, \psi) \in U} (?\varphi; a; ?\psi) \right)_{a \in \mathbf{AP}}$$

Semantics for **DLRC**

Given any $\mathfrak{M} = (W, (R_a)_{a \in \mathbf{AP}}, V)$ & any $w \in W$,

$$\begin{array}{lll}
\mathfrak{M}, w \models [\alpha]\varphi & \text{iff} & \mathfrak{M}, v \models \varphi \text{ for all } v \text{ with } wR_\alpha v, \\
wR_{\alpha \cup \beta} v & \text{iff} & wR_\alpha v \text{ or } wR_\beta v, \\
wR_{\alpha;\beta} v & \text{iff} & wR_\alpha u \text{ and } uR_\beta v \text{ for some } u \in W, \\
wR_{?\varphi} v & \text{iff} & w = v \text{ and } \mathfrak{M}, v \models \varphi, \\
\mathfrak{M}, w \models [\mathsf{r}]\varphi & \text{iff} & \mathfrak{M}^{\mathsf{r}}, w \models \varphi,
\end{array}$$

where $\mathfrak{M}^{\mathsf{r}} = (W, (R^{\mathsf{r}}_a)_{a \in \mathsf{AP}}, V)$ and

$$R^{\mathsf{r}}_a := R_{\mathsf{r}(a)} \text{ where recall that } \mathsf{r}(a) \text{ is a program.}$$

van Benthem & Liu 2007

There is a complete axiomatization of **DLRC**.

Outline

Action Model for Public Ann. of φ

$$\mathbb{E} = (\mathtt{E}, (\mathtt{Q}_a)_{a \in \mathtt{AP}}, \mathtt{pre})$$

- Let $\mathtt{E} = \{\, \mathtt{e} \,\}$, $\mathtt{Q}_a = \{\, (\mathtt{e}, \mathtt{e}) \,\}$, and $\mathtt{pre}(\mathtt{e}) = \varphi$.

What is a single event "action model" of one relation changer r, say, preference upgrade $\mathsf{r}_{\sharp\varphi}$?

Action Model for Preference Upgrade

$$\mathsf{r}_{\sharp\varphi} := (a := (?\neg\varphi; a) \cup (a; ?\varphi))_{a \in \mathbf{AP}}$$

$$\mathbb{E} = (\{\, \mathrm{e} \,\}, (\{\, (\mathrm{e}, \mathrm{e}) \,\})_{a \in \mathtt{AP}}, \boxed{?_1}, \boxed{?_2})$$

- $\boxed{?_1}$: a generalization of `pre` should include both φ and $\neg\varphi$.
- $\boxed{?_2}$ should capture program constructions $\mathsf{r}_{\sharp\varphi}(a)$.

Action Model for Preference Upgrade

$$\mathsf{r}_{\sharp\varphi} := (a := (?\neg\varphi; a) \cup (a; ?\varphi))_{a\in\mathbf{AP}}$$

$$\mathbb{E} = (\{\, \mathsf{e} \,\}, (\{\, (\mathsf{e}, \mathsf{e}) \,\})_{a\in\mathtt{AP}}, \boxed{?_1}, \boxed{?_2})$$

- $\boxed{?_1}$: a generalization of $\mathtt{pre}$ should include both φ and $\neg\varphi$. $\rightsquigarrow \mathtt{CND}(\mathsf{e}) = (\varphi, \neg\varphi)$.
- $\boxed{?_2}$ should capture program constructions $\mathsf{r}_{\sharp\varphi}(a)$. $\rightsquigarrow \mathtt{rel}_a(\mathsf{e}, \mathsf{e}) = (\neg\varphi; a) \cup (a; \varphi)$.

Modified Action Model

$$\mathbb{E} = (\mathtt{E}, (\mathtt{Q}_a)_{a \in \mathtt{AP}}, (\mathtt{CND}(\mathtt{e}))_{\mathtt{e} \in \mathtt{E}}, (\mathtt{rel}_a)_{a \in \mathtt{AP}})$$

- $\mathtt{CND}(\mathtt{e})$: a finite list of formulas (generalization of $\mathtt{pre}$ conditions).
- $\mathtt{rel}_a$: sends each $(\mathtt{e}, \mathtt{f}) \in \mathtt{Q}_a$ to a program $\alpha_{(\mathtt{e},\mathtt{f})}$ s.t.

$$\alpha_{(\mathtt{e},\mathtt{f})} ::= b \mid ?\varphi_{\mathtt{e}} \mid ?\varphi_{\mathtt{f}} \mid (\alpha \cup \alpha) \mid (\alpha ; \alpha),$$

where $b \in \mathbf{AP}$, $\varphi_{\mathtt{e}} \in \mathtt{CND}(\mathtt{e})$, & $\varphi_{\mathtt{f}} \in \mathtt{CND}(\mathtt{f})$.

We also use $\alpha_{(a,\mathtt{e},\mathtt{f})}$ to mean $\mathtt{rel}_a(\mathtt{e}, \mathtt{f})$.

Syntax for **PRC**

Language $\mathcal{L}^{\otimes}$ for **PRC**
(= iteration-free **PDL** + modified action models $\mathbb{E}$):

$$\begin{aligned}
\mathrm{FORM}_{\mathcal{L}^{\otimes}} \ni \varphi &::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid [\alpha]\varphi \mid [\mathbb{E}, \mathsf{e}]\varphi, \\
\mathrm{PR}_{\mathcal{L}^{\otimes}} \ni \alpha &::= a \mid (\alpha \cup \alpha) \mid (\alpha; \alpha) \mid ?\varphi, \\
\mathrm{AM}_{\mathcal{L}^{\otimes}} \ni \mathbb{E} &::= (\mathtt{E}, (\mathtt{Q}_a)_{a \in \mathtt{AP}}, (\mathtt{CND}(\mathtt{e}))_{\mathtt{e} \in \mathtt{E}}, (\mathtt{rel}_a)_{a \in \mathtt{AP}}),
\end{aligned}$$

where $p \in \mathbf{P}$, $a \in \mathbf{AP}$ and $\mathbb{E}$ is a modified action model.

Semantics for **PRC**

Given any $\mathfrak{M} = (W, (R_a)_{a \in \mathbf{AP}}, V)$ & any $w \in W$,

$$\mathfrak{M}, w \models [\mathbb{E}, \mathtt{e}]\varphi \quad \text{iff} \quad \mathfrak{M}^{\otimes \mathbb{E}}, (w, \mathtt{e}) \models \varphi,$$

where $\mathfrak{M}^{\otimes \mathbb{E}} = (W^{\otimes \mathbb{E}}, (R_a^{\otimes \mathbb{E}})_{a \in \mathrm{AP}}, V^{\otimes \mathbb{E}})$ and

$$\begin{array}{rcl} W^{\otimes \mathbb{E}} & := & W \times \mathtt{E}, \\ (w, \mathtt{e}) R_a^{\otimes \mathbb{E}} (v, \mathtt{f}) & \text{iff} & w R_{\alpha_{(a,\mathtt{e},\mathtt{f})}} v \text{ and } \mathtt{e} \mathtt{Q}_a \mathtt{f}, \\ (w, \mathtt{e}) \in V^{\otimes \mathbb{E}}(p) & \text{iff} & w \in V(p), \end{array}$$

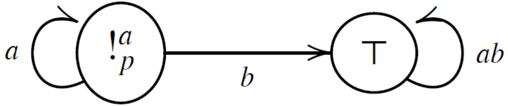
recall that $\alpha_{(a,\mathtt{e},\mathtt{f})} := \mathtt{rel}_a(\mathtt{e}, \mathtt{f})$.

Private Announcement $[!^a_\varphi]$

"*a* alone was privately told that *p*."

Let **AP** $= \{ a, b \}$.

$\mathbb{E}^{!^a_p}$

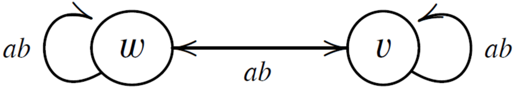


- $\mathtt{CND}(!^a_p) := p$ and $\mathtt{CND}(\top) := \top$,
- $\mathtt{rel}_c(\mathtt{e}, \mathtt{f}) := c; ?\mathtt{CND}(\mathtt{f})$ for all $(\mathtt{e}, \mathtt{f}) \in \mathtt{Q}_c$.

$\not\models [a]p$

$\not\models [b]p$

$\mathfrak{M}$

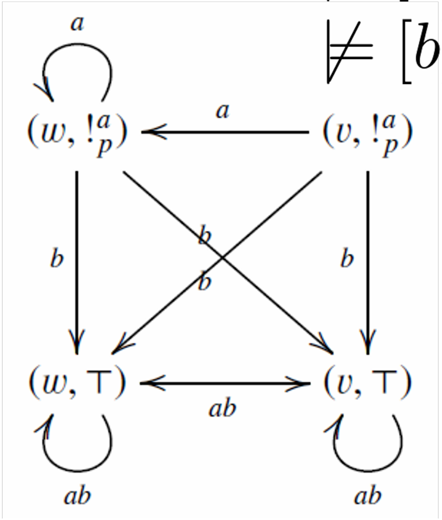


$\models p$ $\qquad$ $\not\models p$

$$\mathfrak{M}\otimes\mathbb{E}^{!^a_p}$$

$$\models [a]p$$

$$\not\models [b]p$$



Hilbert System H**PRC**

Hilbert system of iteration-free **PDL** (Harel et al. 2000) and:

$([\mathbb{E},\mathsf{e}]\mathbf{at})$	$[\mathbb{E},\mathsf{e}]p \leftrightarrow p$
$([\mathbb{E},\mathsf{e}]\neg)$	$[\mathbb{E},\mathsf{e}]\neg\varphi \leftrightarrow \neg[\mathbb{E},\mathsf{e}]\varphi$
$([\mathbb{E},\mathsf{e}]\to)$	$[\mathbb{E},\mathsf{e}](\varphi \to \psi) \leftrightarrow ([\mathbb{E},\mathsf{e}]\varphi \to [\mathbb{E},\mathsf{e}]\psi)$
$([\mathbb{E},\mathsf{e}][a])$	$[\mathbb{E},\mathsf{e}][a]\varphi \leftrightarrow \bigwedge_{\mathtt{f}\in\mathtt{Q}_a(\mathtt{e})}[\alpha_{(a,\mathtt{e},\mathtt{f})}][\mathbb{E},\mathtt{f}]\varphi$
$([\mathbb{E},\mathsf{e}][\cup])$	$[\mathbb{E},\mathsf{e}][\alpha\cup\beta]\varphi \leftrightarrow [\mathbb{E},\mathsf{e}][\alpha]\varphi \wedge [\mathbb{E},\mathsf{e}][\beta]\varphi$
$([\mathbb{E},\mathsf{e}][;])$	$[\mathbb{E},\mathsf{e}][\alpha;\beta]\varphi \leftrightarrow [\mathbb{E},\mathsf{e}][\alpha][\beta]\varphi$
$([\mathbb{E},\mathsf{e}][?])$	$[\mathbb{E},\mathsf{e}][?\psi]\varphi \leftrightarrow [\mathbb{E},\mathsf{e}](\psi \to \varphi)$
$(\mathtt{Nec}_{[\mathbb{E},\mathsf{e}]})$	From φ, infer $[\mathbb{E},\mathsf{e}]\varphi$

We can also derive recursion axioms for **GAUL**.

Completeness of H**PRC**

Theorem

φ is a theorem of H**PRC** iff φ is valid on all models.

($\because$) By reducing the completeness of H**PRC**
to that of iteration-free **PDL** via recursion axioms.

Outline

1. Dynamic Logic of Relation Changers (**DLRC**)
2. Dynamic Logic of Product Relation Changers (**PRC**)
3. Labelled Sequent Calculus for **PRC**

Labelled Formalism for **PDL** (Maffezioli & Naibo 2013)

VAR $= \{x, y, \ldots\}$: a countably infinite set of variables.

$$x : \varphi$$

"A formula φ holds at state x"

$$x\mathrm{R}_\alpha y$$

"There is an α-link from x to y"
(α : program)

$$x = y$$

"State x equals state y"

Labelled Formalism for **PRC**

VAR $= \{x, y, \dots\}$: a countably infinite set of variables.

$$x :^{((\mathbb{E}_1, \mathtt{e}_1), \dots, (\mathbb{E}_n, \mathtt{e}_n))} \varphi$$

$$(x, ((\mathbb{E}_1, \mathtt{e}_1), \dots, (\mathbb{E}_n, \mathtt{e}_n)))\mathrm{R}_\alpha(y, ((\mathbb{E}_1, \mathtt{f}_1), \dots, (\mathbb{E}_n, \mathtt{f}_n)))$$

$$x = y$$

Labelled Sequent Calculus G**PRC**

A sequent is a pair (Γ, Δ) of

finite multisets of labelled expressions in **PRC**.

$$\Gamma \Rightarrow \Delta$$

"if all of Γ hold, then some of Δ holds"

Based on these definitions,
we define rules of labelled sequent calculus G**PRC**.

Features of G**PRC**

- It can be regarded as a natural generalization of G**DLRC** for **DLRC**. (H. et al., 2016)
- Equality rules are adopted from (Seligman, 2001) for hybrid logic.
- The following are key theorems:

If φ is a theorem of G**PRC**, then φ is valid on all models.

All theorems of H**PRC** are also theorems of G**PRC**.

Cut-elimination holds for G**PRC**.

Conclusion

“**PAL**”-style	**AUL** (Kooi & Renne, 2011)	**DLRC** (van Benthem & Liu, 2007)
“**DEL**”-style	**GAUL** (Kooi & Renne, 2011)	Our work!

Main Theorem

TFAE:

1. φ is valid on all models,
2. φ is a theorem of H**PRC**,
3. φ is a theorem of G**PRC**,
4. φ is a theorem of G**PRC** w/o (Cut).

Further Directions

- Explore relationship between General Dynamic Dynamic Logic (Girard et al., 2012) and **PRC**.
- Investigate an alternative semantics of **PRC** (cf. update universe by (van Benthem, 2014)).
- Constructive generalization of **PRC**.

Thank you!